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A TRANSIENT TWO-FLUID MODEL FOR THE SIMULATION OF SLUG FLOW IN PIPELINES—I. THEORY

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Abstract—A one-dimensional transient two-fluid model is developed to predict transient slug flow in pipelines. To account for the interphase interactions, new constitutive relations for the drag coefficient and the virtual mass force for the slug flow regime are derived by applying the conservation equations to a geometrically simplified slug unit. New coefficients in the pressure gradient term in the two-fluid momentum conservation equations are also obtained to account for the non-uniform distribution of the phases and of the pressure drop along a slug unit. The new relations yield a more accurate treatment of the hydrodynamics of slug flow than traditional two-fluid models. Constitutive relations for other flow regimes can also be incorporated into the model, allowing the analysis of general transient two-phase flows in pipelines.

Key Words: gas–liquid flow, two-fluid models, slug flow, transient, pipeline, drag coefficient, virtual mass force, constitutive relations

1. INTRODUCTION

The use of two-phase flow pipelines is common practice in today's petroleum industry. The presence of gas and oil flowing simultaneously in a pipeline can give rise, under certain flow conditions, to important transient problems. One such problem is terrain-induced slugging, which is observed in hilly-terrain pipelines. The liquid phase, being heavier than the gas phase, can accumulate in the valleys to form long liquid bridges that are eventually blown out from one pipeline section to the next due to the gas pressure. This results in a complex flow transient, with significant fluctuations in the outlet liquid mass flow rate. In order to size the gas–liquid separation unit located at the outlet of the pipeline and to determine safe operating conditions for a given pipeline design, it is necessary to be able to analyse such transient two-phase flow problems.

Transient two-fluid models are often used as a tool to predict complex two-phase flows in pipes. Because these models are based on the basic conservation principles for each phase, and treat the interphase interactions at a fundamental level, they can be applied to a wide range of two-phase flow problems. The main difficulty with two-fluid models resides in the development of the constitutive relations which describe the wall to fluid and the interphase interactions through mass, momentum and heat transfer. In the case of the slug flow regime, which is usually the dominant flow pattern in the uphill sections of a pipeline, there exists no satisfactory treatment of the constitutive relations for transient two-fluid models.

Most of the transient two-fluid models are found in the nuclear industry (e.g. RELAP5, Ransom 1983; ATHENA, Richards *et al.* 1985) where the main interest is in predicting fast transients such as in loss of coolant accidents in nuclear power plants. In those models, the slug flow regime constitutive relations are often approximated by correlations used for other flow regimes or by a weighted average, based on void fraction, of the limiting values for the stratified and dispersed flow regimes. Accurate predictions of transient slug flow are, however, not possible without including in the model some details of the slug flow hydrodynamic structure. In recent years, due to problems such as terrain-induced slugging, several transient two-fluid models were developed to analyse pipeline flows (e.g. Fabre *et al.* 1989; Bendiksen *et al.* 1991). In the Fabre *et al.* model, the slug flow regime is represented by a statistically determined sequence in time and space of the stratified

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and dispersed flow regimes. This leads to a complex model for which closure relationships are still under development. The Bendiksen *et al.* model, called OLGA, uses empirical correlations to model slug flow. In addition, it neglects the virtual mass force that arises from the relative acceleration of the phases. This force can be significant for some transient slug flow problems (De Henau 1992).

Other transient models such as the simplified two-fluid model of Taitel *et al.* (1989) or the severe slugging models of Taitel *et al.* (1990) and Sarica *et al.* (1991) have limitations in terms of their application to terrain-induced slugging or general transient slug flow problems.

In this paper, a transient, isothermal, two-fluid model, based on the one-dimensional mass and momentum conservation equations for each phase, is presented. Because there is no adequate treatment of the slug flow regime for two-fluid models, new constitutive relations are derived to account for the interphase momentum interactions. This model is developed with the aim of predicting terrain-induced slugging problems. Because the model is based on basic conservation principles, it is not, however, restricted to terrain-induced slugging and can be used to solve general transient slug flow problems in pipelines. Constitutive relations for stratified, annular and dispersed flows can also be incorporated into the model, allowing the solution of general transient two-phase flows in pipelines.

2. THE BASIC GOVERNING EQUATIONS

The two-fluid model is based on the conservation of mass and momentum for the gas and the liquid phases. In this study, the two-phase flow in a pipeline is assumed to be isothermal, eliminating the need for the energy equation. The basic conservation equations are derived in the work of Mathers *et al.* (1978) where the instantaneous differential equations for mass and momentum for phase k ($k = \mathrm{G}$ for the gas phase, $k = \mathrm{L}$ for the liquid phase) are integrated over an elemental volume occupied instantaneously by phase k, as shown in figure 1. By taking the limit $\Delta x \to 0$, area averaged equations are obtained as:

mass conservation for the gas and liquid phases

$$A\frac{\partial}{\partial t}(\epsilon\rho_{\mathrm{G}}) + \frac{\partial}{\partial x}(A\epsilon\rho_{\mathrm{G}}u_{\mathrm{G}}) = 0$$

$$A\frac{\partial}{\partial t}((1-\epsilon)\rho_{\mathrm{L}}) + \frac{\partial}{\partial x}(A(1-\epsilon)\rho_{\mathrm{L}}u_{\mathrm{L}}) = 0 \qquad [1]$$

momentum conservation in the axial direction (or x-direction) for the gas and liquid phases

$$A\frac{\partial}{\partial t}(\epsilon\rho_{\mathrm{G}}u_{\mathrm{G}}) + \frac{\partial}{\partial x}(A\epsilon\rho_{\mathrm{G}}u_{\mathrm{G}}u_{\mathrm{G}}) + A\epsilon\frac{\partial P_{\mathrm{G}}}{\partial x} + A\epsilon\rho_{\mathrm{G}}g\sin\theta = \Gamma_{\mathrm{Gw}} + \Gamma_{\mathrm{Gi}}$$

$$A\frac{\partial}{\partial t}((1-\epsilon)\rho_{\mathrm{L}}u_{\mathrm{L}}) + \frac{\partial}{\partial x}(A(1-\epsilon)\rho_{\mathrm{L}}u_{\mathrm{L}}u_{\mathrm{L}}) + A(1-\epsilon)\frac{\partial P_{\mathrm{L}}}{\partial x} + A(1-\epsilon)\rho_{\mathrm{L}}g\sin\theta = \Gamma_{\mathrm{Lw}} + \Gamma_{\mathrm{Li}}. \qquad [2]$$

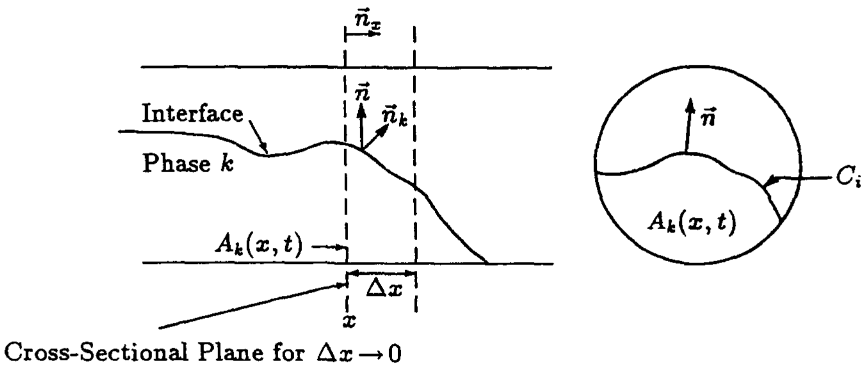

Figure 1. Definitions for the cross-sectional average.

In [1] and [2], A is the cross-sectional area of the pipe and ρ_k, u_k and P_k are, respectively, the density, the velocity and the bulk pressure of phase k. ϵ is the fraction of the area A occupied by the gas phase. For a uniform distribution of the gas phase in the elemental control volume in figure 1, ϵ corresponds to the void fraction. θ is the angle of inclination of the pipe with respect to the horizontal, Γ_{kw} represents the wall shear force per unit length acting on phase k while Γ_{ki} accounts for the x-component of the force per unit length exerted by the interface on phase k through the pressure and the shear stress. The difference between the interface pressure for phase k and the average phase pressure P_k, which is flow regime dependent (the hydrostatic pressure difference in stratified flow for example), is included in the Γ_{ki} term.

In figure 1, $\mathbf{n}_x$ is the unit vector in the axial direction, $\mathbf{n}_k$ is the interface normal unit vector directed away from phase k and $\mathbf{n}$ is the unit vector normal to the line of intersection of the interface and the cross-sectional plane and lying in the cross-sectional plane.

Since [1] and [2] are valid only within each phase, conservation equations are required for the interface. For no mass transfer between the phases, only a momentum balance is needed. By assuming that the interface has a zero thickness and therefore has no mass and no momentum, the x-momentum interfacial conservation equation derived by Mathers *et al.* (1978) reduces to:

$$\langle \mathbf{n}_x \cdot \mathbf{n}_G (P_{Gi} - P_{Li} \rangle_i - \langle \mathbf{n}_x \cdot (\mathbf{n}_G \cdot (\tau_{Gi} - \tau_{Li})) \rangle_i = \left\langle \frac{2\sigma}{R} \mathbf{n}_G \cdot \mathbf{n}_x \right\rangle_i. \qquad [3]$$

The first term of [3] represents the net pressure force acting on the interface while the second term is the net interface shear stress, σ is the surface tension and R represents the mean radius of curvature of the interface and is taken as positive in the direction of $\mathbf{n}_G$. The braces in [3] define an integration along C_i which is the line of intersection between the interface and the cross-sectional plane. For any variable ξ, this integration is given as:

$$\langle \xi \rangle_i = \int_{C_i} \xi \frac{\mathrm{d}C}{\mathbf{n}_k \cdot \mathbf{n}}.$$

An additional equation required to complete the model is the equation of state that relates the gas phase density to the pressure by:

$$\rho_G = \frac{P_G}{\mathscr{R}T}. \qquad [4]$$

Equation [4] assumes that the gas phase behaves as a perfect gas. For an isothermal flow, the temperature T is constant and $\mathscr{R}$ is the gas constant. The liquid density is assumed to be known. It is noted that, in real oil–gas systems, the gas phase generally does not behave as a perfect gas and the liquid phase properties can vary with temperature and pressure. Correlations to account for these effects can, however, be added to the present model.

Equations [1] and [2] are equations for u_G, u_L, ϵ and a cross-sectional average pressure P. To solve these equations requires [3] and [4], as well as additional constitutive relations for Γ_{kw}, Γ_{ki}, P_k, P_{ki}, τ_{ki} that are flow regime dependent. The derivation of these relations is the topic of the next section.

3. THE CONSTITUTIVE RELATIONS

The required constitutive relations for slug flow are developed in this section under the assumptions that the flow is isothermal and that there is no mass transfer between the phases. These relations are obtained by analysing a simplified slug flow "submodel" that uses as input the values of u_G, u_L and ϵ from the solution of [1] and [2]. The submodel provides the constitutive relations and the parameters that arise in these closure relations. The submodel is therefore analogous to a turbulence model that uses the mean-flow variables predicted by the Navier–Stokes equations, and in turn provides the constitutive relations for the Reynolds' stress closure of the mean-flow equations.

The use of a slug flow submodel to provide constitutive relations for the mean-flow equations permits either steady or transient slug flows to be calculated. Standard slug flow models, like the one of Kokal & Stanislav (1989), allow only the solution to steady-state fully developed slug flows.

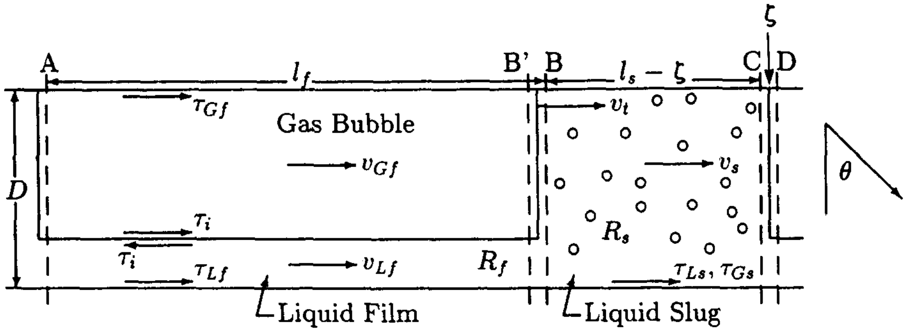

Figure 2. Description of a slug unit.

The submodel has several components that are based on earlier work; these are presented in sections 3.1 and 3.2. The major contributions of the present paper are the development of the drag coefficient, C_D (derived in section 3.3), the derivation of the virtual mass force (section 3.4) and a correction to the pressure gradient term that accounts for void distribution (derived in appendix B and used in section 3.3).

3.1. *Approximations and fundamental relations*

In this section, the approximations used in the slug flow submodel are presented. Based on these approximations, relations for the interfacial and the wall shear stresses for the slug flow regime are obtained.

The simplified geometry of a "slug unit" used in the submodel is illustrated in figure 2. The following approximations are adopted:

SL1—the contribution of the surface tension in the interface momentum balance is assumed to be negligible.

SL2—the phase pressures are assumed to be equal at a given axial location

$$P_L = P_G = P \qquad [5]$$

where the liquid phase is taken as the continuous phase. P is the average pressure over the cross-sectional area A.

SL3—the gas bubbles in the liquid slug portion of a slug unit are uniformly distributed.

SL4—the gas bubbles and the liquid phase in the liquid slug portion of a slug unit have the same velocity.

SL5—the liquid film in the film region of a slug unit is approximated by a uniform liquid layer.

SL6—the Basset forces are ignored.

Following the work of Dukler & Hubbard (1975), the slug unit, which is moving with a translational velocity v_t, is divided into two regions: the "liquid slug region" and "film region", as shown in figure 2. The liquid slug region, of length l_s, is aerated by dispersed bubbles. At the nose of the slug, the bubbles are distributed in the cross-sectional area because of the mixing resulting from the absorption of the liquid film into the liquid slug. However, due to buoyancy effects, the gas bubbles have a tendency to accumulate in the upper portion of the pipe towards the tail of the slug. In the present work, no attempt is made to model the detailed motion of the gas bubbles within the liquid slug and assumption SL3 is used. From assumption SL4, the bubbles have the same velocity as the liquid in the liquid slug zone, which is given by the mixture velocity v_s. This is a good approximation for horizontal and near horizontal pipes. As the inclination angle increases, a drift velocity resulting from buoyancy effects should be added to the mixture velocity (Taitel & Barnea 1990b). The liquid volume fraction in the liquid slug zone is designated by R_s.

The film region, of length l_f, is made up of a liquid film and an elongated bubble. The liquid velocity in the film is denoted by v_{Lf} while the gas velocity in the elongated bubble is v_{Gf}, the liquid fraction is R_f. From assumption SL5, a step change is assumed in the liquid height from the liquid slug to the film region, with the film being approximated by a uniform layer. This is a simplification of the more realistic geometry used for example by Dukler & Hubbard (1975) or by Kokal & Stanislav (1989), in which the liquid height decreases gradually. Assumption SL5 represents a good approximation for a sufficiently long film. A study by Taitel & Barnea (1990a) reveals that the use of the geometry shown in figure 2 yields pressure gradients for steady-state slug flow that are not much different from the pressure gradients obtained when the shape of the liquid film is accounted for. Tests in that study were run under a wide range of flow conditions. Similar conclusions were reached by De Henau (1992).

The main contribution to the pressure drop in slug flow results from the pressure drop in the liquid slug zone. The pressure drop in the film zone becomes signficant only for a long film. It should be added that assumption SL2 is not strictly valid in the film region because of the hydrostatic difference between the liquid layer and the gas bubble above it. However, because the film is assumed to be uniform, this assumption contributes no error to the pressure drop along the film region.

With assumption SL1, the interface momentum equation [3] is rewriten as:

$$\langle \mathbf{n}_x \cdot \mathbf{n}_G (P_{Gi} - P_{Li}) \rangle_i - \langle \mathbf{n}_x \cdot (\mathbf{n}_G \cdot (\tau_{Gi} - \tau_{Li})) \rangle_i = 0. \qquad [6]$$

Equation [6] simply states that the force exerted on the interface by the liquid phase is equal in magnitude and opposite in sign to the force exerted on the interface by the gas phase. The interfacial force per unit length acting on the liquid, Γ_{Li}, is therefore related to the interfacial force per unit length for the gas phase, Γ_{Gi}, by:

$$\Gamma_{Li} = -\Gamma_{Gi}. \qquad [7]$$

Equation [7] can therefore replace [3]. The interfacial force may be assumed to be composed of the drag force, the virtual mass force and the Basset force (Ishii & Mishima 1984). Ignoring the Basset force (approximation SL6), the interfacial force per unit length for the liquid phase in a slug flow regime is written in a form analogous to that of a bubbly flow, that is:

$$\Gamma_{Li} = \tfrac{1}{2} C_D \rho_L |u_r| u_r \frac{\epsilon}{l} A + C_{VM} A \rho_L \frac{du_r^+}{dt}. \qquad [8]$$

u_r is the average relative velocity between the gas and the liquid phase and u_r^+ is a different relative velocity which will be defined later; l is the total length of the slug unit, $l = l_s + l_f$. C_D represents the drag coefficient while C_{VM} is the virtual mass force coefficient.

The wall shear force includes the contributions from the liquid slug zone (denoted by the subscript ks) and from the film zone (subscript kf). The wall shear force per unit length, Γ_{kw}, can therefore be written as:

$$\Gamma_{kw} = \tau_k S_k = \tau_{ks} \frac{l_s}{l} + \tau_{kf} S_{kf} \frac{l_f}{l}, \qquad [9]$$

where τ_k and S_k represent the wall shear stress for phase k and the contact perimeter between phase k and the wall, respectively.

Closure relations are needed for C_D, C_{VM}, du_r^+/dt and for the components of the wall shear force. Also required are the various additional parameters that arise in these closure relations. These parameters are obtained from a detailed analysis of the slug unit illustrated in figure 2.

3.2. Detailed description of the slug flow submodel

As will become evident, the variables required from the submodel are v_{Lf}, v_{Gf}, v_s, v_t, R_s, R_f, l_s, l_f and $u_{r(ss)}$, the average relative velocity between the gas phase and the liquid phase for a steady-state fully developed slug flow. In the development of the submodel, it is assumed that the variables u_G, u_L and ϵ are known from the solution of [1] and [2].

For a slug unit with a uniform film shown in figure 2, the average liquid area fraction $(1-\epsilon)$ is given by:

$$(1-\epsilon) = R_f \frac{l_f}{l} + R_s \frac{l_s}{l}. \qquad [10]$$

A liquid mass balance over a slug unit can be performed by integrating the liquid mass flow rate at a fixed cross-section over the time of passage of a slug unit, as described by Taitel & Barnea (1990b). From this balance, a relationship for v_{Lf} is obtained as:

$$v_{Lf} = \frac{(1-\epsilon)u_L l - R_s v_s l_s}{R_f l_f}. \qquad [11]$$

Using a similar approach for the gas phase, v_{Gf} is given by:

$$v_{Gf} = \frac{\epsilon u_G l - (1-R_s)v_s l_s}{(1-R_f)l_f}. \qquad [12]$$

Assuming that within a slug unit the gas density is constant, a continuity balance on the liquid and the gas indicates that the total volumetric flow rate is independent of axial location. Hence, within the liquid slug zone:

$$v_s = (1-\epsilon)u_L + \epsilon u_G. \qquad [13]$$

The slug unit translational velocity v_t is usually expressed as:

$$v_t = C_0 v_s + v_d, \qquad [14]$$

where C_0 is a weighted flow distribution parameter and v_d is a drift velocity. The following correlations, developed by Théron (1989) for air–water slug flow, are used:

$$C_0 = 1.3 - \frac{0.23}{1 + \left(\frac{\mathrm{Fr}}{\mathrm{Fr_c}}\cos\theta\right)^{10}} + 0.13\sin^2\theta \qquad [15]$$

$$v_d = \sqrt{gD}\left[\left(-0.5 + \frac{0.8}{1 + \left(\frac{\mathrm{Fr}}{\mathrm{Fr_c}}\cos\theta\right)^{10}}\right)\cos\theta + 0.35\sin\theta\right] \qquad [16]$$

with $\mathrm{Fr} = v_s/\sqrt{gD}$, $\mathrm{Fr_c} = 3.5$ and θ is the angle of inclination of the pipe with respect to the horizontal (figure 2).

The liquid fraction within the liquid slug region, R_s, is usually determined by an empirical or semi-empirical correlation. In the present work, the correlation of Andreussi & Bendiksen (1989) is used; this correlation, which is based on a physical model for the production and transport processes for the bubbles in the liquid slug, is:

$$R_s = 1 - \frac{v_s - v_{mf}}{v_s + v_{mo}} \qquad [17]$$

with

$$v_{mf} = 2.6\left[1 - 2\left(\frac{D_0}{D}\right)^2\right]\sqrt{gD} \quad D_0 = 2.5\ \mathrm{cm}$$

$$v_{mo} = 2400[1 - \tfrac{1}{3}\sin\theta]\mathrm{Bo}^{-3/4}\sqrt{gD}$$

$$\mathrm{Bo} = \frac{\rho_L g D^2}{\sigma}$$

D being the pipe diameter and g the gravitational acceleration. An evaluation of various R_s correlations conducted by Théron (1989) shows that the model of Andreussi & Bendiksen (1989) is the most accurate over a wide range of experimental conditions.

The liquid fraction in the liquid film zone, R_f, is calculated from the solution of the liquid phase and gas phase momentum balances for the case of a uniform liquid layer. Eliminating the pressure gradient by combining the liquid and the gas momentum equations yields (Taitel & Barnea 1990b):

$$-(\rho_L - \rho_G)g\sin\theta + \frac{\tau_{Lf}S_{Lf}}{AR_f} - \frac{\tau_{Gf}S_{Gf}}{A(1-R_f)} - \frac{\tau_i S_i}{A}\left(\frac{1}{R_f} + \frac{1}{1-R_f}\right) = 0. \qquad [18]$$

τ_i and S_i are the interface shear stress and width, respectively, in the film region. Since S_{kf} and S_i in [18] depend only on R_f and since the stresses (relations given later) depend on velocities given by [11] and [12] that also are a function of R_f only (for given u_G, u_L and ϵ), [18] can be numerically solved for R_f.

At the present time, the methods to evaluate the length of the liquid slug region, l_s, are still very approximative. Following the recommendations of Kokal & Stanislav (1989) and Taitel & Barnea (1990a), l_s is estimated by:

$$l_s = 30D. \tag{19}$$

For large diameter pipelines ($D > 0.30$ m), Scott *et al.* (1989) show that liquid slugs tend to grow as they flow through the pipeline to lengths larger than 30 pipe diameters. This effect is not accounted for in the present work although the total length l of a slug unit can vary with time.

An additional useful variable supplied by the submodel is the average relative velocity between the gas phase and the liquid phase for a steady-state fully developed slug flow, $u_{r(ss)}$. A liquid mass balance at the nose of the liquid slug (region C–D in figure 2) yields:

$$v_{Lf} R_f = v_t R_f + (v_s - v_t) R_s \tag{20}$$

Combining [10], [11] and [20] gives:

$$u_L = v_t - \frac{R_s}{(1-\epsilon)}(v_t - v_s). \tag{21}$$

Equation [21] is valid only for a steady-state fully developed slug flow. Through a mass balance for the gas phase in region C–D, a similar equation is obtained for u_G:

$$u_G = v_t - \frac{(1 - R_s)}{\epsilon}(v_t - v_s) \tag{22}$$

which yields the relative velocity $u_{r(ss)}$:

$$u_{r(ss)} = u_G - u_L = \frac{R_s - (1-\epsilon)}{\epsilon(1-\epsilon)}(v_t - v_s). \tag{23}$$

Finally, the following correlations are used to evaluate the shear forces in the liquid slug zone and in the film zone:

$$\tau_{Lf} = -\tfrac{1}{2} f_{Lf} \rho_L |v_{Lf}| v_{LF} \quad \tau_{Gf} = -\tfrac{1}{2} f_{Gf} \rho_G |v_{Gf}| v_{Gf} \tag{24}$$

$$\tau_{Ls} = -\tfrac{1}{2} f_{Ls} \rho_L |v_s| v_s \quad \tau_{Gs} = -\tfrac{1}{2} f_{Gs} \rho_G |v_s| v_s \tag{25}$$

$$\tau_i = -\tfrac{1}{2} f_i \rho_G |v_{Gf} - v_{Lf}|(v_{Gf} - v_{Lf}) \tag{26}$$

$$R_f = \frac{\beta_f - \sin\beta_f}{2\pi} \tag{27}$$

$$S_{Lf} = \frac{D}{2}\beta_f \quad S_{Gf} = \frac{D}{2}(2\pi - \beta_f) \tag{28}$$

$$S_{Ls} = \pi D R_s \quad S_{Gs} = \pi D(1 - R_s) \tag{29}$$

$$S_i = D \sin \beta_f/2 \tag{30}$$

β_f is the angle subtended by the liquid wetted perimeter in the film region. The wall and the interface friction factors can be evaluated by a variety of correlations. The correlations used in this paper are given in appendix A.

With u_G, u_L and ϵ given, as already described, v_s is evaluated from [13] and v_t from [14]. l_s is given by [19] and R_s by [17]. v_{Lf}, v_{Gf}, l_f and R_f are calculated by solving numerically [10], [11], [12] and [18]. Equation [23] is used to evaluate $u_{r(ss)}$. These variables describe the slug unit, but do not provide the Γ_{Li} and Γ_{Gi} relations required to close the mean-flow equations ([7] and [8]). This closure is achieved by deriving new relations for C_D, C_{VM} and u_r^+ ([8]), that depend on the submodel just described. The following two sections provide this derivation.

3.3. *Derivation of the slug flow drag coefficient* C_D

An approach often utilized to determine the interface friction factor or drag coefficient in a two-phase pipe flow is to perform an experiment and deduce the interface shear stress or the drag from pressure drop and liquid fraction measurements and the steady-state fully developed momentum equation for either the gas or the liquid phase [see, for example, Andritsos & Hanratty (1987) for interface friction factor for stratified flow]. Although obtained from steady-state data, the experimental correlations for the interface friction factor are used in transient two-fluid models with a reasonable degree of accuracy.

In the present work, it is proposed to derive the slug flow drag coefficient through a similar approach except that the experimental data are replaced by the slug flow submodel. Considering a steady-state fully developed slug flow, the x-momentum conservation equation for the liquid and the gas phases are, from [2]:

$$-A(1-\epsilon)\frac{\partial P_{\mathrm{L}}}{\partial x} - A(1-\epsilon)\rho_{\mathrm{L}}\mathbf{g}\sin\theta + \Gamma_{\mathrm{Lw}} + \Gamma_{\mathrm{Li}} = 0 \qquad [31]$$

$$-A\epsilon\frac{\partial P_{\mathrm{G}}}{\partial x} - A\epsilon\rho_{\mathrm{G}}\mathbf{g}\sin\theta + \Gamma_{\mathrm{Gw}} + \Gamma_{\mathrm{Gi}} = 0. \qquad [32]$$

For a steady-state fully developed slug flow, the interface interaction terms, based on [7] and [8], reduce to:

$$\Gamma_{\mathrm{Li}} = \tfrac{1}{2}C_{\mathrm{D}}\rho_{\mathrm{L}}|u_{\mathrm{r(ss)}}|u_{\mathrm{r(ss)}}\frac{\epsilon A}{l} \quad \Gamma_{\mathrm{Gi}} = -\tfrac{1}{2}C_{\mathrm{D}}\rho_{\mathrm{L}}|u_{\mathrm{r(ss)}}|u_{\mathrm{r(ss)}}\frac{\epsilon A}{l}. \qquad [33]$$

Making use of assumption SL2 and substituting [9] and [33] into [31] and [32] yields:

$$-A(1-\epsilon)\frac{\partial P}{\partial x} - A(1-\epsilon)\rho_{\mathrm{L}}\mathbf{g}\sin\theta + \tau_{\mathrm{L}}S_{\mathrm{L}} + \tfrac{1}{2}C_{\mathrm{D}}\rho_{\mathrm{L}}|u_{\mathrm{r(ss)}}|u_{\mathrm{r(ss)}}\frac{\epsilon A}{l} = 0 \qquad [34]$$

$$-A\epsilon\frac{\partial P}{\partial x} - A\epsilon\rho_{\mathrm{G}}\mathbf{g}\sin\theta + \tau_{\mathrm{G}}S_{\mathrm{G}} - \tfrac{1}{2}C_{\mathrm{D}}\rho_{\mathrm{L}}|u_{\mathrm{r(ss)}}|u_{\mathrm{r(ss)}}\frac{\epsilon A}{l} = 0. \qquad [35]$$

For a pipe with a constant cross-sectional area A, the integration of [34] and [35] over the length of a slug unit gives:

$$-\int_0^l (1-\epsilon)\frac{\partial P}{\partial x}\,\mathrm{d}x - (1-\epsilon)\rho_{\mathrm{L}}l\mathbf{g}\sin\theta + \frac{\tau_{\mathrm{L}}S_{\mathrm{L}}l}{A} + \tfrac{1}{2}C_{\mathrm{D}}\rho_{\mathrm{L}}|u_{\mathrm{r(ss)}}|u_{\mathrm{r(ss)}}\epsilon = 0 \qquad [36]$$

$$-\int_0^l \epsilon\frac{\partial P}{\partial x}\,\mathrm{d}x - \epsilon\rho_{\mathrm{G}}l\mathbf{g}\sin\theta + \frac{\tau_{\mathrm{G}}S_{\mathrm{G}}l}{A} - \tfrac{1}{2}C_{\mathrm{D}}\rho_{\mathrm{L}}|u_{\mathrm{r(ss)}}|u_{\mathrm{r(ss)}}\epsilon = 0. \qquad [37]$$

Because the local area fraction for phase k and the pressure gradient vary along the length of a slug unit, the integral of the pressure terms in [36] and [37] needs special treatment. Denoting the variables at a given cross-section in the slug unit with "*", one may write:

$$\epsilon = \frac{1}{l}\int_0^l \epsilon^*\,\mathrm{d}x$$

$$\frac{\partial P}{\partial x} = \frac{1}{l}\int_0^l \left(\frac{\partial P}{\partial x}\right)^* \mathrm{d}x$$

and

$$\int_0^l \epsilon^*\left(\frac{\partial P}{\partial x}\right)^* \mathrm{d}x = c_{\mathrm{G}}\epsilon\frac{\partial P}{\partial x}l$$

Equations [36] and [37] may then be rewritten as:

$$-c_{\rm L}(1-\epsilon)\frac{\partial P}{\partial x}l-(1-\epsilon)\rho_{\rm L}lg\sin\theta+\frac{\tau_{\rm L}S_{\rm L}l}{A}+\tfrac{1}{2}C_{\rm D}\rho_{\rm L}|u_{\rm r(ss)}|u_{\rm r(ss)}\epsilon=0 \qquad [38]$$

$$-c_{\rm G}\epsilon\frac{\partial P}{\partial x}l-\epsilon\rho_{\rm G}lg\sin\theta+\frac{\tau_{\rm G}S_{\rm G}l}{A}-\tfrac{1}{2}C_{\rm D}\rho_{\rm L}|u_{\rm r(ss)}|u_{\rm r(ss)}\epsilon=0 \qquad [39]$$

with:

$$c_{\rm L}=\frac{\int_0^l(1-\epsilon^*)\left(\frac{\partial P}{\partial x}\right)^*{\rm d}x}{(1-\epsilon)\int_0^l\left(\frac{\partial P}{\partial x}\right)^*{\rm d}x} \qquad [40]$$

$$c_{\rm G}=\frac{\int_0^l\epsilon^*\left(\frac{\partial P}{\partial x}\right)^*{\rm d}x}{\epsilon\int_0^l\left(\frac{\partial P}{\partial x}\right)^*{\rm d}x}=\frac{1-c_{\rm L}(1-\epsilon)}{\epsilon}. \qquad [41]$$

Equations [38] and [39] therefore represent the momentum conservation equations for the liquid and the gas phases, for a steady-state fully developed slug flow, integrated over the length of a slug unit. In those equations, $\partial P/\partial x$ is the average pressure gradient over the length of a slug unit. The wall shear force $\tau_k S_k l$, evaluated by [9], and the area fraction ϵ are also taken as averages over the length of a slug unit. The coefficients $c_{\rm L}$ and $c_{\rm G}$ account for the fact that the liquid and the gas phases are not distributed uniformly along the slug unit and that the pressure gradient in the liquid slug zone is different from the pressure gradient in the film zone. In previous two-fluid models, these coefficients have always been assumed to be unity; $c_{\rm L}$ and $c_{\rm G}$ represent therefore a new component in the two-fluid model for the slug flow regime. The derivation of the $c_{\rm L}$ coefficient is given in appendix B.

By combining [38] and [39], the pressure drop term $(\partial P/\partial x)l$ can be eliminated and an expression for the drag coefficient $C_{\rm D}$ is obtained as:

$$C_{\rm D}=\frac{2c_{\rm L}c_{\rm G}\epsilon(1-\epsilon)}{\rho_{\rm L}\epsilon u^2_{\rm r(ss)}}\left[-\frac{\tau_{\rm L}S_{\rm L}l}{Ac_{\rm L}(1-\epsilon)}+\frac{\tau_{\rm G}S_{\rm G}l}{Ac_{\rm G}\epsilon}+lg\sin\theta\left(\frac{\rho_{\rm L}}{c_{\rm L}}-\frac{\rho_{\rm G}}{c_{\rm G}}\right)\right]. \qquad [42]$$

Equation [42] is the new relationship for the drag coefficient for the slug flow regime. The approach used to derive $C_{\rm D}$ guarantees that for a steady-state fully developed slug flow, the relative velocity $u_{\rm r}$ obtained from the solution of the two-fluid model will be equal to the steady-state relative velocity $u_{\rm r(ss)}$. Except for the average area fraction ϵ and the phase velocity u_k, which come from the mean flow equations, all the variables required to calculate $C_{\rm D}$ from [42] are supplied by the slug flow submodel.

3.4. Derivation of the slug flow virtual mass force

The virtual mass force arises when there is a relative acceleration between the gas and the liquid. Ishii & Mishima (1984) developed a correlation for the virtual mass force for slug flow from a simple potential flow analysis using a Bernouilli equation. In the present study, a similar analysis is used to derive the virtual mass force for the large bubble in the film zone. Because the small gas bubbles in the liquid slug zone are assumed to have the same velocity as the liquid, the contribution to the virtual mass force from that region is ignored.

Referring to figure 2, the mechanical energy equation for the liquid in the control volume A–B′ moving at a velocity $v_{\rm t}$ may be written as:

$$\frac{1}{2}m\frac{\rm d}{{\rm d}t}[(v_{\rm Lf}-v_{\rm t})^2]=\frac{\dot m}{\rho_{\rm L}}(P_{\rm A}-P_{\rm B'}) \qquad [43]$$

where $m=\rho_{\rm L}AR_{\rm f}l_{\rm f}$, $\dot m=\rho_{\rm L}AR_{\rm f}(v_{\rm Lf}-v_{\rm t})$ and $P_{\rm A}$ and $P_{\rm B'}$ are the average pressures over the cross-sectional area at A and B′, respectively. In [43], it is assumed that the mass of the liquid, the internal energy and the potential energy in the control volume are constant.

Simplifying [43] yields:

$$P_A - P_{B'} = -\rho_L l_f \frac{d}{dt}(v_t - v_{Lf}).$$

The force acting on the gas bubble due to an acceleration of the bubble with respect to the liquid film is therefore given by:

$$(P_A - P_{B'})(1 - R_f)A = -\rho_L l_f (1 - R_f) A \frac{d}{dt}(v_t - v_{Lf}). \quad [44]$$

Equation [44] indicates that a positive relative acceleration results in a negative force on the bubble, as expected. The virtual mass force per unit slug length is therefore, from [44]:

$$C_{VM} A \rho_L \frac{du_r^+}{dt} = \frac{l_f}{l}(1 - R_f) A \rho_L \frac{d}{dt}(v_t - v_{lf}) \quad [45]$$

from which the following can be written:

$$C_{VM} = \frac{l_f}{l}(1 - R_f) \quad [46]$$

$$\frac{du_r^+}{dt} = \frac{d}{dt}(v_t - v_{Lf}). \quad [47]$$

The virtual mass coefficient from [46] corresponds to the volume fraction of the large bubble within the slug unit. Ishii & Mishima (1984) obtained a similar virtual mass coefficient, indicating that as the volume fraction of the bubble increases, the virtual mass coefficient should increase due to stronger coupling between the phases.

By combining [20] and [23], $v_t - v_{Lf}$ can be expressed as:

$$v_t - v_{Lf} = \frac{R_s}{R_f} \frac{\epsilon(1-\epsilon)}{R_s - (1-\epsilon)} (u_G - u_L). \quad [48]$$

Although [48] is strictly valid for a steady-state fully developed slug flow, it is used here as an approximation to $v_t - v_{Lf}$ for a general slug flow situation. Substituting [48] into [47] allows writing the virtual mass force per unit slug length as:

$$C_{VM} A \rho_L \frac{du_r^+}{dt} = C_{VM} A \rho_L \left[\frac{R_s}{R_f} \frac{\epsilon(1-\epsilon)}{R_s - (1-\epsilon)} \frac{d}{dt}(u_G - u_L) - \frac{R_s}{R_f} \left(\frac{(2\epsilon - 1)(R_s - (1-\epsilon)) + \epsilon(1-\epsilon)}{(R_s - (1-\epsilon))^2} \right) (u_G - u_L) \frac{d}{dt}(\epsilon) \right]. \quad [49]$$

The total derivative of [49] is defined as:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_t \frac{\partial}{\partial x}.$$

In the derivation of [49], the derivatives of R_s and R_f are not accounted for because these cannot be easily defined in terms of the primitive variables ϵ and u_k. An attempt to do so resulted in very unstable solutions. Hence, for the present time, the virtual mass force is defined by [49] keeping in mind that it is only an approximation.

The virtual mass force derived by Ishii & Mishima (1984) is only the first term of [49] with, however, a different coefficient in front of the total derivative of the relative velocity. The second term in [49] indicates that a change in the liquid area fraction contributes to the virtual mass force since it induces a change in the relative velocity of the phases.

4. MODEL SUMMARY

The one-dimensional transient two-fluid model for slug flow is therefore composed of [1], [2], [4] and [7]. The additional four equations required to close the model are:

- [5] for the phase pressures P_k
- [8] for the interfacial force per unit length for the liquid phase Γ_{Li} with C_D evaluated by [42] and the virtual mass force by [49]
- [9] for the wall shear force for the liquid and the gas phase, Γ_{Lw} and Γ_{Gw}. The components of the wall shear force are given by [24], [25] and [28], [29].

5. CONCLUSIONS

A transient, isothermal, two-fluid model is developed to predict transient slug flow in pipelines. The model is based on the one-dimensional form of the mass and momentum conservation equations and accounts for the wall to fluid and the interphase interactions through constitutive relations. Because there exists no satisfactory treatment of the slug flow regime for two-fluid models, new constitutive relations for the drag coefficient and for the virtual mass force are derived by applying the conservation equations to a geometrically simplified slug unit. New coefficients in the pressure gradient term in the two-fluid momentum conservation equations are also derived to account for the non-uniform distribution of the liquid phase and the local pressure gradient along a slug unit. The resulting two-fluid model can be used to solve general steady-state or transient slug flow problems and has a more accurate treatment of the hydrodynamics of the slug flow regime than traditional transient two-fluid models. While the model theory is discussed in the present paper, comparisons between the model predictions and experimental data for steady-state and transient slug flow problems are presented in the companion paper (De Henau & Raithby 1995).

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APPENDIX A

Evaluation of the Friction Factors

The wall friction factors in the liquid slug and in the film regions are evaluated using:

$$f_k = \begin{cases} \dfrac{16}{\mathrm{Re}_{\mathrm{H}k}} & \mathrm{Re}_{\mathrm{H}k} \leqslant 2000 \\ \left[-3.6\log\left(\dfrac{6.9}{\mathrm{Re}_{\mathrm{H}k}} + \left(\dfrac{e/D_{\mathrm{H}k}}{3.7}\right)^{1.11}\right)\right]^{-2} & \mathrm{Re}_{\mathrm{H}k} \geqslant 3000 \\ \left(3 - \dfrac{6000}{\mathrm{Re}_{\mathrm{H}k}}\right)(f_{\mathrm{turb}} - f_{\mathrm{lam}}) + f_{\mathrm{lam}} & 2000 < \mathrm{Re}_{\mathrm{H}k} < 3000 \end{cases} \qquad [\mathrm{A1}]$$

where e is the pipe wall roughness and $D_{\mathrm{H}k}$ is the hydraulic diameter for phase k. $\mathrm{Re}_{\mathrm{H}k}$ is the Reynolds number based on the hydraulic diameter and is given by:

$$\mathrm{Re}_{\mathrm{H}k} = \frac{\rho_k u_k D_{\mathrm{H}k}}{\eta_k}$$

f_{lam} is f_k for $\mathrm{Re}_{\mathrm{H}k} = 2000$ and f_{turb} is f_k for $\mathrm{Re}_{\mathrm{H}k} = 3000$. The expression for the turbulent wall friction factor follows the recommendation of White (1986) for a single phase turbulent pipe flow.

The liquid slug region is treated as an homogeneous and isotropically distributed mixture, for which case the hydraulic diameters for the gas and the liquid phases are:

$$D_{\mathrm{HL}} = \frac{4A_{\mathrm{L}}}{S_{\mathrm{L}}} = D \quad D_{\mathrm{HG}} = \frac{4A_{\mathrm{G}}}{S_{\mathrm{G}}} = D. \qquad [\mathrm{A2}]$$

The film region is treated as a stratified flow and:

$$D_{\mathrm{HL}} = \frac{4A_{\mathrm{L}}}{S_{\mathrm{L}}} \quad D_{\mathrm{HG}} = \frac{4A_{\mathrm{G}}}{S_{\mathrm{G}} + S_{\mathrm{i}}}. \qquad [\mathrm{A3}]$$

The interface friction factor in the film region, f_{i}, is evaluated using the correlation of Miya *et al.* (1971) for wavy stratified flow:

$$f_{\mathrm{i}} = 0.008 + 2.0 \times 10^{-5}\,\mathrm{Re}_{\mathrm{L}}^{*} \qquad [\mathrm{A4}]$$

with $\mathrm{Re}_{\mathrm{L}}^{*} = \dot{m}_{\mathrm{L}}/\eta_{\mathrm{L}} S_{\mathrm{i}} \cdot \dot{m}_{\mathrm{L}}$ is the liquid mass flow in the film and η_{L} is the liquid viscosity.

APPENDIX B

Derivation of the c_L Coefficient

The expression for c_L is obtained by integrating the liquid and the gas momentum equations over the film zone and the liquid slug zone. For the film zone (from A to B in figure 2), the stratified flow equations for the liquid and the gas phases are used. For the liquid slug zone, a global momentum balance is written for region B–C while region C–D is treated in a similar way as region A–B. The length of regions A–B and B–D are l_f and l_s, respectively. The region C–D has a length ζ where $\zeta \rightarrow 0$.

For a pipe with a constant cross-sectional area, the steady-state momentum equations for the liquid phase and the gas phase, at any cross-section along the length of the slug unit, relative to a coordinate system moving at a velocity v_t, are:

$$\frac{\partial}{\partial x}[(1-\epsilon^*)\rho_L v_L(v_L - v_t)] = -(1-\epsilon^*)\left(\frac{\partial P_L}{\partial x}\right)^* - (1-\epsilon^*)\rho_L g \sin\theta + \frac{\Gamma_{Lw}}{A} + \frac{\Gamma_{Li}}{A} \quad \text{[B1]}$$

$$\frac{\partial}{\partial x}[\epsilon^*\rho_G v_G(v_G - v_t)] = -\epsilon^*\left(\frac{\partial P_G}{\partial x}\right)^* - \epsilon^*\rho_G g \sin\theta + \frac{\Gamma_{Gw}}{A} + \frac{\Gamma_{Gi}}{A} \quad \text{[B2]}$$

where $v_k = v_{kf}$ in the film zone and $v_k = v_s$ in the liquid slug zone. ϵ^* and $(\partial P_k/\partial x)^*$ are local values of the area fraction for the gas and pressure gradient in the slug unit.

The average pressure over a cross-section in the slug unit, P, can be written in terms of P_L and P_G using:

$$P = (1-\epsilon^*)P_L + \epsilon^* P_G. \quad \text{[B3]}$$

It also possible to define the difference between the average interface pressure P_i at any cross-section and the average phase pressure P_k as:

$$\Delta P_{ki} = P_i - P_k. \quad \text{[B4]}$$

Using [B3] and [B4], the following may be derived:

$$P_L = P + \epsilon^*(\Delta P_{Gi} - \Delta P_{Li}) \quad \text{[B5]}$$

$$P_G = P - (1-\epsilon^*)(\Delta P_{Gi} - \Delta P_{Li}). \quad \text{[B6]}$$

For a stratified flow with negligible surface tension, ΔP_{ki} is due only to hydrostatics and is given by:

$$\Delta P_{Li} = -\rho_L g D \cos\theta\left(-\tfrac{1}{2}\cos\frac{\beta}{2} + \frac{1}{3\pi(1-\epsilon^*)}\sin^3\frac{\beta}{2}\right) \quad \text{[B7]}$$

$$\Delta P_{Gi} = +\rho_G g D \cos\theta\left(\tfrac{1}{2}\cos\frac{\beta}{2} + \frac{1}{3\pi\epsilon^*}\sin^3\frac{\beta}{2}\right) \quad \text{[B8]}$$

where $(1-\epsilon^*) = (\beta - \sin\beta)/2\pi$ (β being the angle subtended by the liquid wetted perimeter). For a dispersed flow, the phase pressures can be assumed to be equal (Hancox *et al.* 1980) and:

$$\Delta P_{Li} = \Delta P_{Gi} \quad \text{[B9]}$$

Making use of [B5] and [B7], it can be shown that, for the film region (region A–B), [B1] and [B2] can be rewritten as:

$$\frac{\partial}{\partial x}[(1-\epsilon^*)\rho_L v_L(v_L - v_t)] = -(1-\epsilon^*)\left(\frac{\partial P}{\partial x}\right)^* - (1-\epsilon^*)\frac{\partial}{\partial x}[\epsilon^*(\Delta P_{Gi} - \Delta P_{Li})]$$
$$-(1-\epsilon^*)\rho_L g \sin\theta + \frac{\tau_{Lf}S_{Lf}}{A} - \frac{\tau_i S_i}{A} - \Delta P_{Li}\frac{\partial\epsilon^*}{\partial x} \quad \text{[B10]}$$

$$\frac{\partial}{\partial x}[\epsilon^*\rho_G v_G(v_G - v_t)] = -\epsilon^*\left(\frac{\partial P}{\partial x}\right)^* + \epsilon^*\frac{\partial}{\partial x}[(1-\epsilon^*)(\Delta P_{Gi} - \Delta P_{Li})] - \epsilon^*\rho_G g \sin\theta$$
$$+\frac{\tau_{Gf}S_{Gf}}{A} + \frac{\tau_i S_i}{A} + \Delta P_{Gi}\frac{\partial\epsilon^*}{\partial x} \quad \text{[B11]}$$

with τ_{kf}, τ_i, S_{kf} and S_i defined in appendix A. The terms in [B10] and [B11] that contain ΔP_{Li} and ΔP_{Gi} vary along the film as a function of ϵ^* only. Equations [B10] and [B11] may therefore be rewritten as:

$$\frac{\partial}{\partial x}[(1-\epsilon^*)\rho_L v_L(v_L - v_t)] = -(1-\epsilon^*)\left(\frac{\partial P}{\partial x}\right)^* - (1-\epsilon^*)\rho_L g \sin\theta + \frac{\tau_{Lf}S_{Lf}}{A} - \frac{\tau_i S_i}{A} + F_1(\epsilon^*)\frac{\partial \epsilon^*}{\partial x} \quad \text{[B12]}$$

$$\frac{\partial}{\partial x}[\epsilon^*\rho_G v_G(v_G - v_t)] = -\epsilon^*\left(\frac{\partial P}{\partial x}\right)^* - \epsilon^*\rho_G g \sin\theta + \frac{\tau_{Gf}S_{Gf}}{A} + \frac{\tau_i S_i}{A} + F_2(\epsilon^*)\frac{\partial \epsilon^*}{\partial x} \quad \text{[B13]}$$

$F_1(\epsilon^*)$ and $F_2(\epsilon^*)$ are functions of ϵ^*. For a uniform liquid film, the integration of [B12] over region A–B yields:

$$\int_0^{l_f}(1-\epsilon^*)\left(\frac{\partial P}{\partial x}\right)^* dx = -R_s\rho_L v_s(v_s - v_t) + R_f\rho_L v_{Lf}(v_{Lf} - v_t) - R_f l_f \rho_L g \sin\theta + \frac{\tau_{Lf}S_{Lf}l_f}{A} - \frac{\tau_i S_i l_f}{A} + \int_{R_f}^{R_s} F_1(\epsilon^*)\,d\epsilon^* \quad \text{[B14]}$$

An expression for $\tau_i S_i l_f/A$ may be obtained by combining [B12] and [B13], applied to the region of uniform liquid film, to give:

$$\frac{\tau_i S_i l_f}{A} = \frac{\tau_{Lf}S_{Lf}l_f(1-R_f)}{A} - \frac{\tau_{Gf}S_{Gf}l_f R_f}{A} - R_f(1-R_f)l_f(\rho_L - \rho_G)g\sin\theta \quad \text{[B15]}$$

Substituting [B15] into [B14] gives:

$$\int_0^{l_f}(1-\epsilon^*)\left(\frac{\partial P}{\partial x}\right)^* dx = -R_s\rho_L v_s(v_s - v_t) + R_f\rho_L v_{Lf}(v_{Lf} - v_t) - R_f l_f[\rho_L R_f + \rho_G(1-R_f)]g\sin\theta + \frac{\tau_{Lf}S_{Lf}l_f R_f}{A} + \frac{\tau_{Gf}S_{Gf}l_f R_f}{A} + \int_{R_f}^{R_s} F_1(\epsilon^*)\,d\epsilon^* \quad \text{[B16]}$$

Now, adding [B12] and [B13] results in:

$$\left(\frac{\partial P}{\partial x}\right)^* = -\frac{\partial}{\partial x}[(1-\epsilon^*)\rho_L v_L(v_L - v_t)] - \frac{\partial}{\partial x}[\epsilon^*\rho_G v_G(v_G - v_t)] - (1-\epsilon^*)\rho_L g\sin\theta - \epsilon^*\rho_G g\sin\theta + \frac{\tau_{Lf}S_{Lf}}{A} + \frac{\tau_{Gf}S_{Gf}}{A} + [F_1(\epsilon^*) + F_2(\epsilon^*)]\frac{\partial\epsilon^*}{\partial x} \quad \text{[B17]}$$

Integrating [B17] over region A–B gives:

$$\int_0^{l_f}\left(\frac{\partial P}{\partial x}\right)^* dx = -R_s\rho_L v_s(v_s - v_t) + R_f\rho_L v_{Lf}(v_{Lf} - v_t) - (1-R_s)\rho_G v_s(v_s - v_t) + (1-R_f)\rho_G v_{Gf}(v_{Gf} - v_t) - l_f[\rho_L R_f + \rho_G(1-R_f)]g \times \sin\theta + \frac{\tau_{Lf}S_{Lf}l_f}{A} + \frac{\tau_{Gf}S_{Gf}l_f}{A} + \int_{R_f}^{R_s}[F_1(\epsilon^*) + F_2(\epsilon^*)]d\epsilon^* \quad \text{[B18]}$$

Because the length of region C–D is infinitively small, the momentum conservation for that region reduces to a balance between the momentum fluxes and the pressure forces. The integration of [B12] over that region is therefore:

$$\int_{l_f+l_s-\zeta}^{l_f+l_s}(1-\epsilon^*)\left(\frac{\partial P}{\partial x}\right)^* dx = -R_f\rho_L v_{Lf}(v_{Lf} - v_t) + R_s\rho_L v_s(v_s - v_t) - \int_{R_f}^{R_s} F_1(\epsilon^*)\,d\epsilon^* \quad \text{[B19]}$$

while from [B17]:

$$\int_{l_f+l_s-\zeta}^{l_f+l_s}\left(\frac{\partial P}{\partial x}\right)^{*} \mathrm{d}x = -R_f\rho_L v_{Lf}(v_{Lf}-v_t)+R_s\rho_L v_s(v_s-v_t)-(1-R_f)\rho_G v_{Gf}(v_{Gf}-v_t)$$
$$+(1-R_s)\rho_G v_s(v_s-v_t)-\int_{R_f}^{R_s}[F_1(\epsilon^*)+F_2(\epsilon^*)]\,\mathrm{d}\epsilon^* \quad [\mathrm{B20}]$$

For region B–C, which is treated as a dispersed flow, a global momentum balance is obtained by adding [B1] and [B2]. As a result of [B9], the summation of [B1] and [B2] yields:

$$\left(\frac{\partial P}{\partial x}\right)^{*} = -\frac{\partial}{\partial x}[(1-\epsilon^*)\rho_L v_L(v_L-v_t)]-\frac{\partial}{\partial x}[\epsilon^*\rho_G v_G(v_G-v_t)]-(1-\epsilon^*)\rho_L g$$
$$\times\sin\theta-\epsilon^*\rho_G g\sin\theta+\frac{\tau_{Ls}S_{Ls}}{A}+\frac{\tau_{Gs}S_{Gs}}{A} \quad [\mathrm{B21}]$$

Multiplying [B21] by $(1-\epsilon^*)$ and integrating over region B–C gives:

$$\int_{l_f}^{l_f+l_s-\zeta}(1-\epsilon^*)\left(\frac{\partial P}{\partial x}\right)^{*}\mathrm{d}x = -R_s l_s[\rho_L R_s+\rho_G(1-R_s)]g\sin\theta+\frac{\tau_{Ls}S_{Ls}l_sR_s}{A}+\frac{\tau_{Gs}l_sR_s}{A}. \quad [\mathrm{B22}]$$

It is assumed here that, over region B–C, ϵ^* is constant and equal to $(1-R_s)$. In a similar fashion, from the integration of [B21] over region B–C, one obtains:

$$\int_{l_f}^{l_f+l_s-\zeta}\left(\frac{\partial P}{\partial x}\right)^{*}\mathrm{d}x = -l_s[\rho_L R_s+\rho_G(1-R_s)]g\sin\theta+\frac{\tau_{Ls}S_{Ls}l_s}{A}+\frac{\tau_{Gs}S_{Gs}l_s}{A}. \quad [\mathrm{B23}]$$

Hence, summing [B16], [B19] and [B22] results in:

$$\int_0^l(1-\epsilon^*)\left(\frac{\partial P}{\partial x}\right)^{*}\mathrm{d}x = -R_f l_f[\rho_L R_f+\rho_G(1-R_f)]g\sin\theta-R_s l_s[\rho_L R_s+\rho_G(1-R_s)]g\sin\theta$$
$$+\frac{\tau_{Lf}S_{Lf}l_fR_f}{A}+\frac{\tau_{Gf}S_{Gf}l_fR_f}{A}+\frac{\tau_{Ls}S_{Ls}l_sR_s}{A}+\frac{\tau_{Gs}S_{Gs}l_sR_s}{A} \quad [\mathrm{B24}]$$

while from [B18], [B20] and [B23]:

$$\int_0^l\left(\frac{\partial P}{\partial x}\right)^{*}\mathrm{d}x = -l_f[\rho_L R_f+\rho_G(1-R_f)]g\sin\theta-l_s[\rho_L R_s+\rho_G(1-R_s)]g\sin\theta$$
$$+\frac{\tau_{Lf}S_{Lf}l_f}{A}+\frac{\tau_{Gf}S_{Gf}l_f}{A}+\frac{\tau_{Ls}S_{Ls}l_s}{A}+\frac{\tau_{Gs}S_{Gs}l_s}{A}. \quad [\mathrm{B25}]$$

Equations [B24] and [B25] are used in [40] to obtain c_L.